Nils A. Baas · Eric M. Friedlander
Bjørn Jahren · Paul Arne Østvær
Editors

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ABEL SYMPOSIA 4

Algebraic Topology

The Abel Symposium 2007

Springer

ABEL SYMPOSIA

Edited by the Norwegian Mathematical Society

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Algebraic Topology

The Abel Symposium 2007

Proceedings of the Fourth Abel Symposium,
Oslo, Norway, August 5–10, 2007

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ISBN 978-3-642-01199-3 e-ISBN: 978-3-642-01200-6
DOI: 10.1007/978-3-642-01200-6
Springer Dodrecht Heidelberg London New York

Library of Congress Control Number: 2009926010

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface to the Series

The Niels Henrik Abel Memorial Fund was established by the Norwegian government on January 1, 2002. The main objective is to honor the great Norwegian mathematician Niels Henrik Abel by awarding an international prize for outstanding scientific work in the field of mathematics. The prize shall contribute towards raising the status of mathematics in society and stimulate the interest for science among school children and students. In keeping with this objective the board of the Abel fund has decided to finance one or two Abel Symposia each year. The topic may be selected broadly in the area of pure and applied mathematics. The Symposia should be at the highest international level, and serve to build bridges between the national and international research communities. The Norwegian Mathematical Society is responsible for the events. It has also been decided that the contributions from these Symposia should be presented in a series of proceedings, and Springer Verlag has enthusiastically agreed to publish the series. The board of the Niels Henrik Abel Memorial Fund is confident that the series will be a valuable contribution to the mathematical literature.

Ragnar Winther
Chairman of the board of the Niels Henrik Abel Memorial Fund

Preface

The 2007 Abel Symposium took place at the University of Oslo from August 5 to August 10, 2007. The aim of the symposium was to bring together mathematicians whose research efforts have led to recent advances in algebraic geometry, algebraic K-theory, algebraic topology, and mathematical physics. The common theme of this symposium was the development of new perspectives and new constructions with a categorical flavor. As the lectures at the symposium and the papers of this volume demonstrate, these perspectives and constructions have enabled a broadening of vistas, a synergy between once-differentiated subjects, and solutions to mathematical problems both old and new.

This symposium was organized by two complementary groups: an external organizing committee consisting of Eric Friedlander (Northwestern,University of Southern California), Stefan Schwede (Bonn) and Graeme Segal (Oxford) and a local organizing committee consisting of Nils A. Baas (Trondheim), Bjørn Ian Dundas (Bergen), Bjørn Jahren (Oslo) and John Rognes (Oslo).

The webpage of the symposium can be found at http://abelsymposium.no/symp2007/info.html

The interested reader will find titles and abstracts of the talks listed here

		Monday 6th	Tuesday 7th	Wednesday 8th	Thursday 9th
09.30	10.30	F. Morel	S. Stolz	J. Lurie	N. Strickland
11.00	12.00	M. Hopkins	A. Merkurjev	J. Baez	U. Jannsen
13.30	14.20	R. Cohen	M. Behrens	H. Esnault	C. Rezk
14.40	15.30	L. Hesselholt	M. Rost	B. Toën	M. Levine
16.30	17.20	M. Ando		U. Tillmann	D. Freed
17.40	18.30	D. Sullivan		V. Voevodsky	

as well as a few online lecture notes. Moreover, one will find here useful information about our gracious host city, Oslo.

The present volume consists of 12 papers written by participants (and their collaborators). We give a very brief overview of each of these papers.

"The classifying space of a topological 2-group" by J.C. Baez and D. Stevenson: Recent work in higher gauge theory has revealed the importance of categorising the theory of bundles and considering various notions of 2-bundles. The present paper gives a survey on recent work on such generalized bundles and their classification.

"String topology in dimensions two and three" by M. Chas and D. Sullivan describes some applications of string topology in low dimensions for surfaces and 3-manifolds. The authors relate their results to a theorem of W. Jaco and J. Stallings and a group theoretical statement equivalent to the three-dimensional Poincare conjecture.

The paper "Floer homotopy theory, realizing chain complexes by module spectra, and manifolds with corners", by R. Cohen, extends ideas of the author, J.D.S. Jones and G. Segal on realizing the Floer complex as the cellular complex of a natural stable homotopy type. Crucial in their work was a framing condition on certain moduli spaces, and Cohen shows that by replacing this by a certain orientability with respect to a generalized cohomology theory E^*, there is a natural definition of Floer E_*–homology.

In "Relative Chern characters for nilpotent ideals" by G. Cortiñas and C. Weibel, the equality of two relative Chern characters from K-theory to cyclic homology is shown in the case of nilpotent ideals. This important equality has been assumed without proof in various papers in the past 20 years, including recent investigations of negative K-theory.

H. Esnault's paper "Algebraic differential characters of flat connections with nilpotent residues" shows that characteristic classes of flat bundles on quasi-projective varieties lift canonically to classes over a projective completion if the local monodromy at infinity is unipotent. This should facilitate the computations in some situations, for it is sometimes easier to compute such characteristic classes for flat bundles on quasi-projective varieties.

"Norm varieties and the Chain Lemma (after Markus Rost)" by C. Haesemeyer and C. Weibel gives detailed proofs of two results of Markus Rost known as the "Chain Lemma" and the "Norm principle". The authors place these two results in the context of the overall proof of the Bloch–Kato conjecture. The proofs are based on lectures by Rost.

In "On the Whitehead spectrum of the circle" L. Hesselholt extends the known computations of homotopy groups $\pi_q(\mathrm{Wh}^{\mathrm{Top}}(S^1))$ of the Whitehead spectrum of the circle. This problem is of fundamental importance for understanding the homeomorphism groups of manifolds, in particular those admitting Riemannian metrics of negative curvature. The author achieves complete and explicit computations for $q \leq 3$.

J.F. Jardine's paper "Cocycle categories" presents a new approach to defining and manipulating cocycles in right proper model categories. These cocycle methods provide simple new proofs of homotopy classification results for torsors, abelian sheaf cohomology groups, group extensions and gerbes. It is also shown that the algebraic K-theory presheaf of spaces is a simplicial stack associated to a suitably defined parabolic groupoid.

“A survey of elliptic cohomology” by J. Lurie is an expository account of the relationship between elliptic cohomology and derived algebraic geometry. This paper lies at the intersection of homotopy theory and algebraic geometry. Precursors are kept to a minimum, making the paper accessible to readers with a basic background in algebraic geometry, particularly with the theory of elliptic curves. A more comprehensive account with complete definitions and proofs, will appear elsewhere.

“On Voevodsky’s algebraic K-theory spectrum” by I. Panin, K. Pimenov and O. Röndigs resolves some K-theoretic questions in the modern setting of motivic homotopy theory. They show that the motivic spectrum BGL, which represents algebraic K-theory in the motivic stable homotopy category, has a unique ring structure over the integers. For general base schemes this structure pulls back to give a distinguished monoidal structure which the authors have employed in their proof of a motivic Conner–Floyd theorem.

The paper “Chern character, loop spaces and derived algebraic geometry” authored by B. Toën and G. Vezzosi presents work in progress dealing with a categorised version of sheaf theory. Central to their work is the new notion of “derived categorical sheaves”, which categorises the notion of complexes of sheaves of modules on schemes. Ideas originating in derived algebraic geometry and higher category theory are used to introduce “derived loop spaces” and to construct a Chern character for categorical sheaves with values in cyclic homology. This work can be seen as an attempt to define algebraic analogs of elliptic objects and characteristic classes.

“Voevodsky’s lectures on motivic cohomology 2000/2001” consists of four parts, each dealing with foundational material revolving around the proof of the Bloch–Kato conjecture. A motivic equivariant homotopy theory is introduced with the specific aim of extending non-additive functors, such as symmetric products, from schemes to the motivic homotopy category. The text is written by P. Deligne.

We gratefully acknowledge the generous support of the Board for the Niels Henrik Abel Memorial Fund and the Norwegian Mathematical Society. We also thank Ruth Allewelt and Springer-Verlag for constant encouragement and support during the editing of these proceedings.

Trondheim, Los Angeles and Oslo
March 2009

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The Classifying Space of a Topological 2-Group

John C. Baez and Danny Stevenson*

Abstract Categorifying the concept of topological group, one obtains the notion of a "topological 2-group". This in turn allows a theory of "principal 2-bundles" generalizing the usual theory of principal bundles. It is well known that under mild conditions on a topological group G and a space M, principal G-bundles over M are classified by either the Čech cohomology $\check{H}^1(M,G)$ or the set of homotopy classes $[M,BG]$, where BG is the classifying space of G. Here we review work by Bartels, Jurčo, Baas–Bökstedt–Kro, and others generalizing this result to topological 2-groups and even topological 2-categories. We explain various viewpoints on topological 2-groups and the Čech cohomology $\check{H}^1(M,\mathcal{G})$ with coefficients in a topological 2-group $\mathcal{G}$, also known as "nonabelian cohomology". Then we give an elementary proof that under mild conditions on M and $\mathcal{G}$ there is a bijection $\check{H}^1(M,\mathcal{G}) \cong [M, B|\mathcal{G}|]$ where $B|\mathcal{G}|$ is the classifying space of the geometric realization of the nerve of $\mathcal{G}$. Applying this result to the "string 2-group" String(G) of a simply-connected compact simple Lie group G, it follows that principal String(G)-2-bundles have rational characteristic classes coming from elements of $H^*(BG,\mathbb{Q})/\langle c\rangle$, where c is any generator of $H^4(BG,\mathbb{Q})$.

1 Introduction

Recent work in higher gauge theory has revealed the importance of categorifying the theory of bundles and considering "2-bundles", where the fiber is a topological category instead of a topological space [4]. These structures show up not only in mathematics, where they form a useful generalization of nonabelian gerbes [8], but also in physics, where they can be used to describe parallel transport of strings [29, 30].

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* Danny Stevenson has received support from Collaborative Research Center 676: "Particles, Strings, and the Early Universe".

N.A. Baas et al. (eds.), *Algebraic Topology*, Abel Symposia 4,
DOI 10.1007/978-3-642-01200-6_1, © Springer-Verlag Berlin Heidelberg 2009

The concepts of "Čech cohomology" and "classifying space" play a well-known and fundamental role in the theory of bundles. For any topological group G, principal G-bundles over a space M are classified by the first Čech cohomology of M with coefficients in G. Furthermore, under some mild conditions, these Čech cohomology classes are in 1–1 correspondence with homotopy classes of maps from M to the classifying space BG. This lets us define characteristic classes for bundles, coming from cohomology classes for BG.

All these concepts and results can be generalized from bundles to 2-bundles. Bartels [5] has defined principal $\mathcal{G}$-2-bundles where $\mathcal{G}$ is a "topological 2-group": roughly speaking, a categorified version of a topological group. Furthermore, his work shows how principal $\mathcal{G}$-2-bundles over M are classified by $\check{H}^1(M, \mathcal{G})$, the first Čech cohomology of M with coefficients in $\mathcal{G}$. This form of cohomology, also known as "nonabelian cohomology", is familiar from work on nonabelian gerbes [7, 17].

In fact, under mild conditions on $\mathcal{G}$ and M, there is a 1–1 correspondence between $\check{H}^1(M, \mathcal{G})$ and the set of homotopy classes of maps from M to a certain space $B|\mathcal{G}|$: the classifying space of the geometric realization of the nerve of $\mathcal{G}$. So, $B|\mathcal{G}|$ serves as a classifying space for the topological 2-group $\mathcal{G}$! This paper seeks to provide an introduction to topological 2-groups and nonabelian cohomology leading up to a self-contained proof of this fact.

In his pioneering work on this subject, Jurčo [20] asserted that a certain space homotopy equivalent to ours is a classifying space for the first Čech cohomology with coefficients in $\mathcal{G}$. However, there are some gaps in his argument for this assertion (see Sect. 5.2 for details).

Later, Baas, Bökstedt and Kro [1] gave the definitive treatment of classifying spaces for 2-bundles. For any "good" topological 2-category $\mathcal{C}$, they construct a classifying space $B\mathcal{C}$. They then show that for any space X with the homotopy type of a CW complex, concordance classes of "charted $\mathcal{C}$-2-bundles" correspond to homotopy classes of maps from M to $B\mathcal{C}$. In particular, a topological 2-group is just a topological 2-category with one object and with all morphisms and 2-morphisms invertible – and in this special case, their result *almost* reduces to the fact mentioned above.

There are some subtleties, however. Most importantly, while their "charted $\mathcal{C}$-2-bundles" reduce precisely to our principal $\mathcal{G}$-2-bundles, their classify these 2-bundles up to concordance, while we classify them up to a superficially different equivalence relation. Two $\mathcal{G}$-2-bundles over a space X are "concordant" if they are restrictions of some $\mathcal{G}$-2-bundle over $X \times [0,1]$ to the two ends $X \times \{0\}$ and $X \times \{1\}$. This makes it easy to see that homotopic maps from X to the classifying space define concordant $\mathcal{G}$-2-bundles. We instead consider two $\mathcal{G}$-2-bundles to be equivalent if their defining Čech 1-cocycles are cohomologous. In this approach, some work is required to show that homotopic maps from X to the classifying space define equivalent $\mathcal{G}$-2-bundles. A priori, it is not obvious that two $\mathcal{G}$-2-bundles are equivalent in this Čech sense if and only if they are concordant. However, since the classifying space of Baas, Bökstedt and Kro is homotopy equivalent to the one we use, it follows from that these equivalence relations are the same – at least given $\mathcal{G}$ and M satisfying the technical conditions of both their result and ours.

We also discuss an interesting example: the "string 2-group" String(G) of a simply-connected compact simple Lie group G [2, 18]. As its name suggests, this 2-group is of special interest in physics. Mathematically, a key fact is that $|$String $(G)|$– the geometric realization of the nerve of String(G) – is the 3-connected cover of G. Using this, one can compute the rational cohomology of $B|\text{String}(G)|$. This is nice, because these cohomology classes give "characteristic classes" for principal $\mathcal{G}$-2-bundles, and when M is a manifold one can hope to compute these in terms of a connection and its curvature, much as one does for ordinary principal bundles with a Lie group as structure group.

Section 2 is an overview, starting with a review of the classic results that people are now categorifying. Section 3 reviews four viewpoints on topological 2- groups. Section 4 explains nonabelian cohomology with coefficients in a topological 2-group. Finally, in Sect. 5 we prove the results stated in Sect. 2, and comment a bit further on the work of Jurčo and Baas–Bökstedt–Kro.

2 Overview

Once one knows about "topological 2-groups", it is irresistibly tempting to generalize all ones favorite results about topological groups to these new entities. So, let us begin with a quick review of some classic results about topological groups and their classifying spaces.

Suppose that G is a topological group. The Čech cohomology $\check{H}^1(M, G)$ of a topological space M with coefficients in G is a set carefully designed to be in 1–1 correspondence with the set of isomorphism classes of principal G-bundles on M. Let us recall how this works.

First suppose $\mathcal{U} = \{U_i\}$ is an open cover of M and P is a principal G-bundle over M that is trivial when restricted to each open set U_i. Then by comparing local trivialisations of P over U_i and U_j we can define maps $g_{ij}: U_i \cap U_j \to G$: the transition functions of the bundle. On triple intersections $U_i \cap U_j \cap U_k$, these maps satisfy a cocycle condition:

$$g_{ij}(x)g_{jk}(x) = g_{ik}(x)$$

A collection of maps $g_{ij}: U_i \cap U_j \to G$ satisfying this condition is called a "Čech 1-cocycle" subordinate to the cover $\mathcal{U}$. Any such 1-cocycle defines a principal G-bundle over M that is trivial over each set U_i.

Next, suppose we have two principal G-bundles over M that are trivial over each set U_i, described by Čech 1-cocycles g_{ij} and g'_{ij}, respectively. These bundles are isomorphic if and only if for some maps $f_i: U_i \to G$ we have

$$g_{ij}(x)f_j(x) = f_i(x)g'_{ij}(x)$$

on every double intersection $U_i \cap U_j$. In this case we say the Čech 1-cocycles are "cohomologous". We define $\check{H}^1(\mathcal{U}, G)$ to be the quotient of the set of Čech 1-cocycles subordinate to $\mathcal{U}$ by this equivalence relation.

Recall that a "good" cover of M is an open cover $\mathcal{U}$ for which all the non-empty finite intersections of open sets U_i in $\mathcal{U}$ are contractible. We say a space M *admits good covers* if any cover of M has a good cover that refines it. For example, any (paracompact Hausdorff) smooth manifold admits good covers, as does any simplicial complex.

If M admits good covers, $\check{H}^1(\mathcal{U}, G)$ is independent of the choice of good cover $\mathcal{U}$. So, we can denote it simply by $\check{H}^1(M, G)$. Furthermore, this set $\check{H}^1(M, G)$ is in 1–1 correspondence with the set of isomorphism classes of principal G-bundles over M. The reason is that we can always trivialize any principal G-bundle over the open sets in a good cover.

For more general spaces, we need to define the Čech cohomology more carefully. If M is a paracompact Hausdorff space, we can define it to be the inverse limit

$$\check{H}^1(M, G) = \varprojlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, G)$$

over all open covers, partially ordered by refinement.

It is a classic result in topology that $\check{H}^1(M, G)$ can be understood using homotopy theory with the help of Milnor's construction [13, 26] of the classifying space BG:

Theorem 0. *Let G be a topological group. Then there is a topological space BG with the property that for any paracompact Hausdorff space M, there is a bijection*

$$\check{H}^1(M, G) \cong [M, BG]$$

Here $[X, Y]$ denotes the set of homotopy classes of maps from X into Y. The topological space BG is called the *classifying space* of G. There is a canonical principal G-bundle on BG, called the universal G-bundle, and the theorem above is usually understood as the assertion that every principal G-bundle P on M is obtained by pullback from the universal G-bundle under a certain map $M \to BG$ (the classifying map of P).

Now let us discuss how to generalize all these results to topological 2-groups. First of all, what is a "2-group"? It is like a group, but "categorified". While a group is a *set* equipped with *functions* describing multiplication and inverses, and an identity *element*, a 2-group is a *category* equipped with *functors* describing multiplication and inverses, and an identity *object*. Indeed, 2-groups are also known as "categorical groups".

A down-to-earth way to work with 2-groups involves treating them as "crossed modules". A crossed module consists of a pair of groups H and G, together with a homomorphism $t: H \to G$ and an action α of G on H satisfying two conditions, (4) and (5) below. Crossed modules were introduced by Whitehead [35] without the aid of category theory. Mac Lane and Whitehead [22] later proved that just as the fundamental group captures all the homotopy-invariant information about a connected pointed homotopy 1-type, a crossed module captures all the homotopy-invariant information about a connected pointed homotopy 2-type. By the 1960s it was clear to Verdier and others that crossed modules are essentially the same as categorical groups. The first published proof of this may be due to Brown and Spencer [10].

Just as one can define principal G-bundles over a space M for any topological group G, one can define "principal $\mathcal{G}$-2-bundles" over M for any topological 2-group $\mathcal{G}$. Just as a principal G-bundle has a copy of G as fiber, a principal $\mathcal{G}$-2-bundle has a copy of $\mathcal{G}$ as fiber. Readers interested in more details are urged to read Bartels' thesis, available online [5]. We shall have nothing else to say about principal $\mathcal{G}$-2-bundles except that they are classified by a categorified version of Čech cohomology, denoted $\check{H}^1(M, \mathcal{G})$.

As before, we can describe this categorified Čech cohomology as a set of cocycles modulo an equivalence relation. Let $\mathcal{U}$ be a cover of M. If we think of the 2-group $\mathcal{G}$ in terms of its associated crossed module (G, H, t, α), then a cocycle subordinate to $\mathcal{U}$ consists (in part) of maps $g_{ij} : U_i \cap U_j \to G$ as before. However, we now "weaken" the cocycle condition and only require that

$$t(h_{ijk}) g_{ij} g_{jk} = g_{ik} \tag{1}$$

for some maps $h_{ijk} : U_i \cap U_j \cap U_k \to H$. These maps are in turn required to satisfy a cocycle condition of their own on quadruple intersections, namely

$$\alpha(g_{ij})(h_{jkl}) h_{ijl} = h_{ijk} h_{ikl} \tag{2}$$

where α is the action of G on H. This mildly intimidating equation will be easier to understand when we draw it as a commuting tetrahedron – see (6) in the next section. The pair (g_{ij}, h_{ijk}) is called a $\mathcal{G}$-valued *Čech 1-cocycle* subordinate to $\mathcal{U}$.

Similarly, we say two cocycles (g_{ij}, h_{ijk}) and (g'_{ij}, h'_{ijk}) are *cohomologous* if

$$t(k_{ij}) g_{ij} f_j = f_i g'_{ij} \tag{3}$$

for some maps $f_i : U_i \to G$ and $k_{ij} : U_i \cap U_j \to H$, which must make a certain prism commute – see (7). We define $\check{H}^1(\mathcal{U}, \mathcal{G})$ to be the set of cohomology classes of $\mathcal{G}$-valued Čech 1-cocycles. To capture the entire cohomology set $\check{H}^1(M, \mathcal{G})$, we must next take an inverse limit of the sets $\check{H}^1(\mathcal{U}, \mathcal{G})$ as $\mathcal{U}$ ranges over all covers of M. For more details we refer to Sect. 4.

Theorem 0 generalizes nicely from topological groups to topological 2-groups:

Theorem 1. *Suppose that $\mathcal{G}$ is a well-pointed topological 2-group and M is a paracompact Hausdorff space admitting good covers. Then there is a bijection*

$$\check{H}^1(M, \mathcal{G}) \cong [M, B|\mathcal{G}|]$$

where the topological group $|\mathcal{G}|$ is the geometric realization of the nerve of $\mathcal{G}$.

One term here requires explanation. A topological group G is said to be "well pointed" if $(G, 1)$ is an NDR pair, or in other words if the inclusion $\{1\} \hookrightarrow G$ is a closed cofibration. We say that a topological 2-group $\mathcal{G}$ is *well pointed* if the topological groups G and H in its corresponding crossed module are well pointed. For example, any "Lie 2-group" is well pointed: a topological 2-group is called a *Lie 2-group* if G and H are Lie groups and the maps t, α are smooth. More generally, any "Fréchet Lie 2-group" [2] is well pointed. We explain the importance of this notion in Sect. 5.1.

Bartels [5] has already considered two examples of principal $\mathcal{G}$-2-bundles, corresponding to abelian gerbes and nonabelian gerbes. Let us discuss the classification of these before turning to a third, more novel example.

For an abelian gerbe [11], we first choose an abelian topological group H – in practice, usually just U(1). Then, we form the crossed module with $G = 1$ and this choice of H, with t and α trivial. The corresponding topological 2-group deserves to be called $H[1]$, since it is a "shifted version" of H. Bartels shows that the classification of abelian H-gerbes matches the classification of $H[1]$-2-bundles. It is well known that

$$|H[1]| \simeq BH$$

so the classifying space for abelian H-gerbes is

$$B|H[1]| \simeq B(BH)$$

In the case $H = \mathrm{U}(1)$, this classifying space is just $K(\mathbb{Z}, 3)$. So, in this case, we recover the well-known fact that abelian U(1)-gerbes over M are classified by

$$[M, K(\mathbb{Z}, 3)] \cong H^3(M, \mathbb{Z})$$

just as principal U(1) bundles are classified by $H^2(M, \mathbb{Z})$.

For a nonabelian gerbe [7,16,17], we fix any topological group H. Then we form the crossed module with $G = Aut(H)$ and this choice of H, where $t\colon H \to G$ sends each element of H to the corresponding inner automorphism, and the action of G on H is the tautologous one. This gives a topological 2-group called $\mathrm{AUT}(H)$. Bartels shows that the classification of nonabelian H-gerbes matches the classification of $\mathrm{AUT}(H)$-2-bundles. It follows that, under suitable conditions on H, nonabelian H-gerbes are classified by homotopy classes of maps into $B|\mathrm{AUT}(H)|$.

A third application of Theorem 1 arises when G is a simply-connected compact simple Lie group. For any such group there is an isomorphism $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$ and the generator $\nu \in H^3(G, \mathbb{Z})$ transgresses to a characteristic class $c \in H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$. Associated to ν is a map $G \to K(\mathbb{Z}, 3)$ and it can be shown that the homotopy fiber of this can be given the structure of a topological group $\hat{G}$. This group $\hat{G}$ is the 3-connected cover of G. When $G = \mathrm{Spin}(n)$, this group $\hat{G}$ is known as $\mathrm{String}(n)$. In general, we might call $\hat{G}$ the *string group* of G. Note that until one picks a specific construction for the homotopy fiber, $\hat{G}$ is only defined up to homotopy – or more precisely, up to equivalence of A_∞-spaces.

In [2], under the above hypotheses on G, a topological 2-group subsequently dubbed the *string 2-group* of G was introduced. Let us denote this by $\mathrm{String}(G)$. A key result about $\mathrm{String}(G)$ is that the topological group $|\mathrm{String}(G)|$ is equivalent to $\hat{G}$. By construction $\mathrm{String}(G)$ is a Fréchet Lie 2-group, hence well pointed. So, from Theorem 1 we immediately conclude:

Corollary 1. *Suppose that G is a simply-connected compact simple Lie group. Suppose M is a paracompact Hausdorff space admitting good covers. Then there are bijections between the following sets:*

- *The set of equivalence classes of principal* $\mathrm{String}(G)$*-2-bundles over M*
- *The set of isomorphism classes of principal $\hat{G}$-bundles over M*
- $\check{H}^1(M, \mathrm{String}(G))$
- $\check{H}^1(M, \hat{G})$
- $[M, B\hat{G}]$

One can describe the rational cohomology of $B\hat{G}$ in terms of the rational cohomology of BG, which is well understood. The following result was pointed out to us by Matt Ando (personal communication), and later discussed by Greg Ginot [15]:

Theorem 2. *Suppose that G is a simply-connected compact simple Lie group, and let $\hat{G}$ be the string group of G. Let $c \in H^4(BG, \mathbb{Q}) = \mathbb{Q}$ denote the transgression of the generator $\nu \in H^3(G, \mathbb{Q}) = \mathbb{Q}$. Then there is a ring isomorphism*

$$H^*(B\hat{G}, \mathbb{Q}) \cong H^*(BG, \mathbb{Q})/\langle c \rangle$$

where $\langle c \rangle$ is the ideal generated by c.

As a result, we obtain characteristic classes for $\mathrm{String}(G)$-2-bundles:

Corollary 2. *Suppose that G is a simply-connected compact simple Lie group and M is a paracompact Hausdorff space admitting good covers. Then an equivalence class of principal* $\mathrm{String}(G)$*-2-bundles over M determines a ring homomorphism*

$$H^*(BG, \mathbb{Q})/\langle c \rangle \to H^*(M, \mathbb{Q})$$

To see this, we use Corollary 1 to reinterpret an equivalence class of principal $\mathcal{G}$-2-bundles over M as a homotopy class of maps $f: M \to B|\mathcal{G}|$. Picking any representative f, we obtain a ring homomorphism

$$f^*: H^*(B|\mathcal{G}|, \mathbb{Q}) \to H^*(M, \mathbb{Q}).$$

This is independent of the choice of representative. Then, we use Theorem 2.

It is a nice problem to compute the rational characteristic classes of a principal $\mathrm{String}(G)$-2-bundle over a manifold using de Rham cohomology. It should be possible to do this using the curvature of an arbitrary connection on the 2-bundle, just as for ordinary principal bundles with a Lie group as structure group. Sati et al. [29] have recently made excellent progress on solving this problem and its generalizations to n-bundles for higher n.

3 Topological 2-Groups

In this section we recall four useful perspectives on topological 2-groups. For a more detailed account, we refer the reader to [3].

A *topological* 2-*group* is a groupoid in the category of topological groups. In other words, it is a groupoid $\mathcal{G}$ where the set $\mathrm{Ob}(\mathcal{G})$ of objects and the set $\mathrm{Mor}(\mathcal{G})$ of morphisms are each equipped with the structure of a topological group such that the source and target maps $s,t\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$, the map $i\colon \mathrm{Ob}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G})$ assigning each object its identity morphism, the composition map $\circ\colon \mathrm{Mor}(\mathcal{G}) \times_{\mathrm{Ob}(\mathcal{G})} \mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G})$, and the map sending each morphism to its inverse are all continuous group homomorphisms.

Equivalently, we can think of a topological 2-group as a group in the category of topological groupoids. A *topological groupoid* is a groupoid $\mathcal{G}$ where $\mathrm{Ob}(\mathcal{G})$ and $\mathrm{Mor}(\mathcal{G})$ are topological spaces and all the groupoid operations just listed are continuous maps. We say that a functor $f\colon \mathcal{G} \to \mathcal{G}'$ between topological groupoids is *continuous* if the maps $f\colon \mathrm{Ob}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G}')$ and $f\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G}')$ are continuous. A group in the category of topological groupoids is such a thing equipped with continuous functors $m\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $\mathrm{inv}\colon \mathcal{G} \to \mathcal{G}$ and a unit object $1 \in \mathcal{G}$ satisfying the usual group axioms, written out as commutative diagrams.

This second viewpoint is useful because any topological groupoid $\mathcal{G}$ has a "nerve" $N\mathcal{G}$, a simplicial space where the space of n-simplices consists of composable strings of morphisms

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} x_{n-1} \xrightarrow{f_n} x_n$$

Taking the geometric realization of this nerve, we obtain a topological space which we denote as $|\mathcal{G}|$ for short. If $\mathcal{G}$ is a topological 2-group, its nerve inherits a group structure, so that $N\mathcal{G}$ is a topological simplicial group. This in turn makes $|\mathcal{G}|$ into a topological group. This passage from the topological 2-group $\mathcal{G}$ to the topological group $|\mathcal{G}|$ will be very important in what follows.

A third way to understand topological 2-groups is to view them as topological crossed modules. Recall that a *topological crossed module* (G, H, t, α) consists of topological groups G and H together with a continuous homomorphism

$$t\colon H \to G$$